

Aharonov-Anandan phase in nonunitary cyclic dynamics

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Abstract. The Aharonov-Anandan (AA) phase is an important concept crucial to both fundamental understanding and quantum control applications. Often regarded as a nonadiabatic version of the Berry phase, the AA phase reflects the geometry of a curve traced out by an actual cyclic time evolution. Using the AA phase, this work exposes rich physics in nonunitary cyclic time evolution of non-Hermitian systems that can describe a wide class of systems in the classical domain. We first show that a previous expression for the AA phase, though originally derived for Hermitian systems, can equally apply to non-Hermitian systems so long as the initial state is normalized. This result then indicates that the AA phase in nonunitary dynamics is always real, thus clearing much confusion in the literature. We then analyze the AA phase in two periodically driven non-Hermitian models. In the slow-driving limit, the AA phase reduces to the Berry phase in the first case, but oscillates violently and does not approach any limit in the second case. The rich geometrical features of nonunitary dynamics are thus exposed for the first time, also opening up potential applications of the AA phase in non-Hermitian systems.

Keywords: AA phase, Berry phase, time dependent Hamiltonians, non-Hermitian Hamiltonians, \mathcal{PT} symmetry

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1. Introduction

The importance of geometrical phases in physics has been widely known [1]. While Berry phase reflects the geometry of instantaneous Hamiltonian eigenstates in a projective Hilbert space [2], the Aharonov-Anandan (AA) phase [3] reflects the geometry of a curve in a projective Hilbert space traced out by actual cyclic time evolution. One important application of the AA phase is nonadiabatic holonomic quantum computation [4]. But remarkably, there has been much confusion on geometric phase if it is extended to nonunitary dynamics. Though Samuel and Bhandari generalized Berry phase in the most general setting and obtained always real Berry phases [5], Mostafazadeh proposed a real Berry phase using the dual eigenstates in the biorthonormal basis [6], complex Berry phases or complex AA phases in non-Hermitian systems were obtained or positively discussed in other studies [7, 8, 9, 10, 11, 12, 13].

Spectral and dynamical aspects of non-Hermitian systems have attracted considerable theoretical and experimental interests [14, 15, 16, 17, 18]. Such systems may be regarded as certain extensions of quantum mechanics, but more often they model realistic systems in the classical domain with loss and gain, such as waveguides [19, 20], LRC circuits [21], mechanical oscillators [22, 23, 24, 25, 26], as well as acoustic systems [27, 28]. However, little is known about the dynamics of these systems when it is nonunitary. For this reason and the above-mentioned confusion on geometric phases, it is more necessary than ever to study nonunitary dynamics from a fundamental view.

Non-Hermitian systems, if periodically driven, can produce nonunitary but stable time evolution [29, 30]. This finding makes non-Hermitian systems more useful and quantum mechanics tools more relevant to non-Hermitian systems. In this work we use such systems to illuminate on general features of the AA phase and use the AA phase as a diagnostic tool to investigate the geometrical aspects of nonunitary but cyclic evolution.

Let $|\psi(t)\rangle$ be the time evolving state, which is cyclic as time evolves from $t = 0$ to $t = T$. We shall show that an earlier expression for AA phase (denoted β), originally written down for Hermitian systems (see Eq. (22) in [10]), namely,

$$\beta = \frac{1}{i} \ln \langle \psi(0) | \psi(T) \rangle + i \int_0^T dt \frac{\langle \psi(t) | \dot{\psi}(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}, \quad (1)$$

can equally apply to nonunitary dynamics so long as the initial state is normalized, i.e., $\langle \psi(0) | \psi(0) \rangle = 1$. Though not obvious at the first glance, the above expression for the AA phase (and hence likewise Berry phase in the original meaning) is always real (to be shown below) in arbitrary nonunitary dynamics. This lays a solid foundation in order to use AA phase as a tool to study non-Hermitian systems. Two specific non-Hermitian models under periodic driving are studied as examples. As the driving period increases, the AA phase smoothly approaches the Berry phase in the first case that does show adiabatic following, but violently oscillates and does not have a limit at all in the second case due to an exotic hopping phenomenon. As a largely unexplored problem in physics, the geometrical features of nonunitary dynamics are thus seen to be a fruitful topic for future experimental and theoretical studies.

2. Revisiting AA phase in nonunitary dynamics

2.1. General considerations

To appreciate how Eq. (1) applies to general nonunitary dynamics, we now return to the time-evolving state $|\psi(t)\rangle$ being cyclic at $t = T$. That is,

$$|\psi(T)\rangle = e^{i\alpha}|\psi(0)\rangle. \quad (2)$$

The context of $|\psi(t)\rangle$ and its detailed time dependence (may satisfy a linear or even nonlinear equation of motion) are not needed here. Because the dynamics under consideration is nonunitary in general, α can be complex. The above-defined phase factor α can be expressed as

$$\alpha = \frac{1}{i} \ln \frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(0)|\psi(0)\rangle}. \quad (3)$$

The normalization of $|\psi(t)\rangle$ is of no interest in developing geometrical insights into the dynamics. Indeed, since the geometry in the projective Hilbert space is of the main concern, there is no reason to be particularly interested in the time dependence of the normalization of $|\psi(t)\rangle$. Associated with $|\psi(t)\rangle$, we now define a normalized time-evolving state as

$$|\phi(t)\rangle \equiv \frac{|\psi(t)\rangle}{\sqrt{\langle\psi(t)|\psi(t)\rangle}}. \quad (4)$$

Note again that this renormalization procedure *only* removes the non-norm-preserving aspect of nonunitary dynamics, with other impact of nonunitary dynamics still captured by the renormalized state $|\phi(t)\rangle$. We then have

$$\langle\phi(t)|\dot{\phi}(t)\rangle = \frac{\langle\psi(t)|\dot{\psi}(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} - \frac{1}{2} \frac{d}{dt} \ln \langle\psi(t)|\psi(t)\rangle. \quad (5)$$

Evidently, $\langle\phi(t)|\dot{\phi}(t)\rangle$ is always purely imaginary. Via the construction above, $|\phi(t)\rangle$ is also a cyclic state, with the following property

$$|\phi(T)\rangle = e^{i\text{Re}\alpha}|\phi(0)\rangle. \quad (6)$$

Consider next the following time-evolving state

$$|\varphi(t)\rangle \equiv e^{-if(t)}|\phi(t)\rangle, \quad (7)$$

with the real function $f(t)$ satisfying

$$f(T) - f(0) = \text{Re}\alpha. \quad (8)$$

Clearly,

$$|\varphi(T)\rangle = |\varphi(0)\rangle. \quad (9)$$

So $|\varphi(t)\rangle$ is a periodic function of t , thus being a single-valued function along a closed curve in the projective Hilbert space traced out by $|\psi(t)\rangle$. A connection of $|\varphi(t)\rangle$ along

this closed curve, namely, $\langle \varphi(t) | \dot{\varphi}(t) \rangle$, can be well defined and easily evaluated. One immediately has

$$\langle \varphi(t) | \dot{\varphi}(t) \rangle = -i\dot{f}(t) + \langle \phi(t) | \dot{\phi}(t) \rangle. \quad (10)$$

The AA phase is then obtained as an integral of the connection of $|\varphi(t)\rangle$ along the closed curve in the projective Hilbert space,

$$\begin{aligned} \beta &\equiv i \int_0^T dt \langle \varphi(t) | \dot{\varphi}(t) \rangle \\ &= f(T) - f(0) + i \int_0^T dt \langle \phi(t) | \dot{\phi}(t) \rangle \end{aligned} \quad (11)$$

$$= \frac{1}{i} \ln \frac{\langle \psi(0) | \psi(T) \rangle}{\langle \psi(0) | \psi(0) \rangle} + i \int_0^T dt \frac{\langle \psi(t) | \dot{\psi}(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} \quad (12)$$

$$= \alpha + i \int_0^T dt \frac{\langle \psi(t) | \dot{\psi}(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}. \quad (13)$$

A few important remarks are in order. First, because $\langle \varphi(t) | \dot{\varphi}(t) \rangle$ is purely imaginary as $\langle \phi(t) | \dot{\phi}(t) \rangle$, the AA phase β obtained above is always real, irrespective of the context of the cyclic state $|\psi(t)\rangle$. Second, β is gauge-invariant [10]. That is, multiplying $|\psi(t)\rangle$ by an arbitrary time-dependent c-number factor, one obtains precisely the same AA phase. This further confirms that the AA phase obtained above reflects the geometry of a closed curve in a projective Hilbert space. In addition, it can be also easily checked that β can be understood as a consequence of a parallel transport along this curve. Third, if the cyclic dynamics is known to arise from adiabatic following of some instantaneous eigenstates of some (non-Hermitian) Hamiltonian (see below), then the AA phase will become the Berry phase by definition, and the resulting Berry phase must be always real, too. Indeed, if the cyclic state follows instantaneous energy eigenstates, then they differ only by a dynamical phase factor and the gauge invariance guarantees that the AA phase is the same as the Berry phase. Fourth, the above final expression for AA phase is almost the same as in Ref. [10], with the only difference being the $\langle \psi(0) | \psi(0) \rangle$ factor in the first term of Eq. (12). That is, the earlier expression for AA phase [10] can equally apply to arbitrary nonunitary dynamics so long as $\langle \psi(0) | \psi(0) \rangle = 1$, with Eq. (12) reducing to Eq. (1). This fourth point is one main finding of this work. Interestingly, authors of Ref. [10] did not realize the generality and the always real nature of the β expression above. Instead, they [11] adopted an approach based on biorthonormal basis [7] to tackle nonunitary dynamics. That approach yields complex Berry connection and complex geometric phases in general [7, 11, 12], of which the physics is unclear as compared with the original meaning of geometric phase.

2.2. Nonunitary dynamics in systems with non-Hermitian Hamiltonians

If $|\psi(t)\rangle$ is governed by a Schrödinger equation with a non-Hermitian Hamiltonian $H(t)$,

$$i\hbar|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle, \quad (14)$$

then Eq. (5) becomes

$$\langle \phi(t) | \dot{\phi}(t) \rangle = \frac{1}{2i\hbar} \langle \phi(t) | H(t) + H^\dagger(t) | \phi(t) \rangle. \quad (15)$$

The AA phase obtained in Eq. (11) becomes

$$\begin{aligned} \beta &= \text{Re } \alpha + \frac{1}{2\hbar} \int_0^T dt \langle \phi(t) | H(t) + H^\dagger(t) | \phi(t) \rangle \\ &= \text{Re } \alpha + \frac{1}{\hbar} \text{Re} \int_0^T dt \langle \phi(t) | H(t) | \phi(t) \rangle \\ &= \text{Re } \alpha + \frac{1}{\hbar} \text{Re} \int_0^T dt \frac{\langle \psi(t) | H(t) | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}. \end{aligned} \quad (16)$$

Interestingly, plugging Eq. (14) into Eq. (13), one arrives at an alternative but equivalent expression, namely,

$$\beta = \alpha + \frac{1}{\hbar} \int_0^T dt \frac{\langle \psi(t) | H(t) | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}. \quad (17)$$

However, it should be stressed that the AA phase expressions of Eqs. (11) and (13) are general and Eq. (16) and Eq. (17) only apply to those cases where the time evolution is governed by a Schrödinger equation. Related to this, it is also necessary to discuss the dynamical phase. It is often said that the overall phase of a cyclic state is the sum of a dynamical phase and a geometric phase. Applying this understanding to Eq. (13) or Eq. (17), it is seen that α is the overall phase complex in general, whereas the dynamical phase $-\frac{1}{\hbar} \int_0^T dt \frac{\langle \psi(t) | H(t) | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}$ is also complex in general. On the other hand, Eq. (16) indicates that, in representation of normalized states $|\phi(t)\rangle$, both the overall phase and the dynamical phase are always real. This second perspective is consistent with the one adopted by Samuel and Bhandari [5]. It is important to note that the above two pictures based on $|\psi(t)\rangle$ and $|\phi(t)\rangle$ are equivalent because AA phase is gauge-invariant. The criticisms in Ref. [9] on Ref. [5] are hence unfounded.

3. A solvable model

A periodically driven, non-Hermitian Hamiltonian (in dimensionless units) treated previously [7, 9, 12] is given by

$$H_1(t) = \cos(\omega t) \sigma_x + \sin(\omega t) \sigma_y + \epsilon \sigma_z, \quad (18)$$

where σ_x , σ_y , and σ_z represent the standard Pauli matrices, ω is real and ϵ is complex in general. The propagator $U(t)$ of this system can be calculated analytically and cyclic states after one period $T = 2\pi/\omega$ can be found by solving for the eigenvectors of $U(T)$. There are in general two cyclic states $|u_\pm\rangle$ of $U(T)$, whose explicit time dependence is given by

$$|u_\pm(t)\rangle \equiv U(t)|u_\pm\rangle = e^{\mp i\Omega t - i\frac{1}{2}\omega t + i\gamma_\pm} \begin{pmatrix} \cos\left(\frac{1}{2}\Theta_\pm\right) \\ \sin\left(\frac{1}{2}\Theta_\pm\right) e^{i\Phi_\pm} \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned}\Omega &\equiv \sqrt{1 + \left(\epsilon - \frac{1}{2}\omega\right)^2}, \\ \Theta_{\pm} &= 2 \cot^{-1} \left| \epsilon - \frac{1}{2}\omega \pm \Omega \right|, \\ \Phi_{\pm} &= \omega t - \gamma_{\pm},\end{aligned}\tag{20}$$

and γ_{\pm} is the phase of $\left(\epsilon - \frac{1}{2}\omega \pm \Omega\right)$.

With Eq. (19), Eq. (16) yields the AA phases β_{\pm} for $|u_{\pm}(t)\rangle$, i.e.,

$$\beta_{\pm} = \frac{2\pi \left| \epsilon - \frac{1}{2}\omega \pm \Omega \right|^2}{\left| \epsilon - \frac{1}{2}\omega \pm \Omega \right|^2 + 1}.\tag{21}$$

Clearly, AA phases in Eq. (21) are nothing but the half of the solid angles traced out by the cyclic states $|u_{\pm}(t)\rangle$ on the Bloch sphere, i.e.,

$$\beta_{\pm} = \pi (1 + \cos \Theta_{\pm}).\tag{22}$$

It is curious to examine what happens in the slow-driving limit, namely, cases with $\omega \rightarrow 0$. First of all, the AA phases in Eq. (21) reduce to

$$\beta_{\pm} \rightarrow \frac{2\pi \left| \epsilon \pm \sqrt{1 + \epsilon^2} \right|^2}{\left| \epsilon \pm \sqrt{1 + \epsilon^2} \right|^2 + 1}.\tag{23}$$

On the other hand, cyclic states $|u_{\pm}(t)\rangle$ are found to reduce to instantaneous energy eigenstates of $H_1(t)$ up to overall phases. This indicates that β_{\pm} obtained above become Berry phases and they are certainly real. By contrast, the Berry phase obtained in Ref. [7] in our notation would be $\beta_{\pm}^{\text{GW}} = \pi \left(1 - \frac{\epsilon}{\sqrt{1+\epsilon^2}}\right)$, which is complex in general and does not have a geometrical interpretation as the original Berry phase.

4. A “hopping” model

Here we study a second simple model with the following non-Hermitian Hamiltonian (in dimensionless units)

$$H_2(t) = \sigma_z + i\mu(\cos \omega t + i)\sigma_x.\tag{24}$$

As previously shown by us [30], though $H_2(t)$ is non-Hermitian, its time evolution can be stable because its cyclic states may only acquire a real overall phase after one period. Our results discussed below are indeed in this stable region. Analytical cyclic states here are not available. We hence use two numerically obtained cyclic states as the initial states. We then analyze the details of the resulting cyclic time evolution in terms of the ratio of the two components of the time-evolving state, as compared with the instantaneous eigenstates of $H_2(t)$. Of particular interest here is what happens if the driving period $T = 2\pi/\omega$ becomes large.

For small values of μ , it is found that the cyclic states can follow the instantaneous eigenstates of $H_2(t)$ as the driving slows down. As illustrated in Fig. 1, for T of the order of hundreds, adiabatic following of the cyclic states with the $H_2(t)$ eigenstates is already clearly visible. Indeed, the AA phase obtained numerically approaches a zero geometric

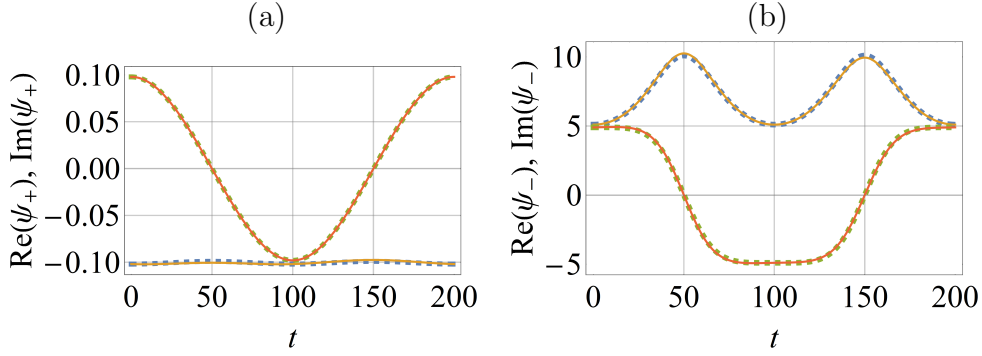


Figure 1. (color online) Comparison between the cyclic states of the model Eq. (24) for $\mu = 0.2$ and the instantaneous eigenstates of $H_2(t)$ for sufficiently slow driving ($T = 200$). The plotted vertical coordinates represent the time dependence of the real and imaginary parts of the ratios (denoted ψ_{\pm}) of the two components of the time-evolving cyclic states (solid lines) or the eigenstates of $H_2(t)$ (dotted line), compared with two respective instantaneous eigenstates of $H_2(t)$. Panels (a) and (b) are for two different cyclic states. That the solid lines almost perfectly overlap with dotted lines indicates adiabatic following.

phase, in agreement with a direct calculation (based on instantaneous eigenstates of $H_2(t)$) that gives a zero Berry phase.

However, as shown in Fig. 2, for larger values of μ , the above observations break down completely. In cases of slow driving, the time evolution of the cyclic states now displays exotic dynamics by *hopping* between two instantaneous eigenstates. Before and after one hopping, a cyclic state tends to follow one of the instantaneous eigenstates of $H_2(t)$. This suggests that when the overall time evolution is stable, local instability can still dominate over the dynamics during certain small time windows.

The hopping phenomenon here clearly demonstrates that adiabatic following in nonunitary dynamics may not hold. However, in previous studies [31, 32], one typically uses an instantaneous eigenstate of the Hamiltonian as the initial state, and then adiabatic following may break down due to non-negligible accumulation of non-adiabatic transitions in nonunitary dynamics. By contrast, here we instead use a cyclic state as the initial state (which is very close to, but not the same as instantaneous eigenstates of the Hamiltonian for slow driving). Now even though the time evolution is both stable and cyclic, the dynamics still displays intriguing hopping and hence violates adiabatic following. This hints that the breakdown of adiabatic following observed here is on a different (and perhaps more fundamental) level than studied before [32]. Indeed, if we also use cyclic states as the initial states for the model studied in Ref. [32], then we would obtain perfect adiabatic following under slow driving. The hopping phenomenon discovered here, whose underlying mathematical structure is yet to be better understood, is another main finding of this work and also constitutes a motivator towards further exploration of cyclic nonunitary dynamics.

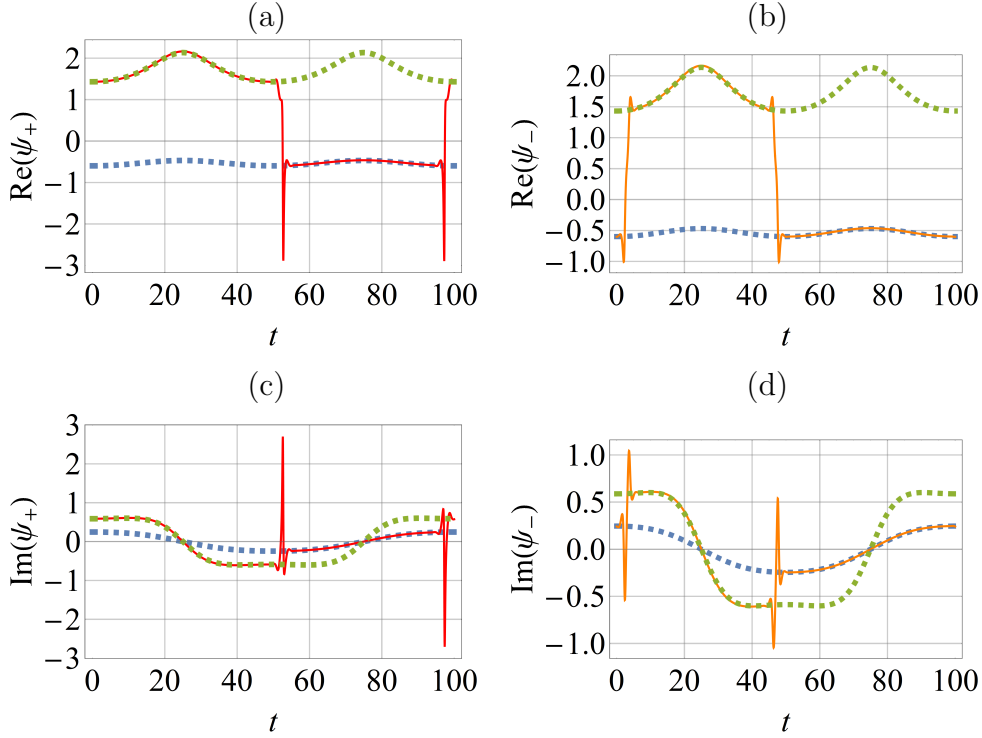


Figure 2. (color online) Comparison between two cyclic states of the model depicted by Eq. (24) for $\mu = 1.2$ and the two instantaneous eigenstates of $H_2(t)$ for sufficiently slow driving ($T = 100$). The plotted vertical coordinates represent the ratios (denoted ψ_{\pm}) of the two components of time-evolving cyclic states (solid lines), as compared with the parallel behavior of two instantaneous eigenstates of $H_2(t)$ (upper and lower dotted lines). Panels (a) and (c) are for one cyclic state, and panels (b) and (d) are for the other cyclic state.

Figure 3 presents on the Bloch sphere the exotic hopping dynamics of one cyclic state shown in Fig. 2. It is seen that, due to the hopping between two instantaneous eigenstates, the geometry of the curve traced out by the cyclic state becomes highly nontrivial. Specifically, in this example each instantaneous eigenstate of $H_2(t)$ does not trace out a solid angle on the Bloch sphere (indicating zero Berry phase), but the cyclic state does trace out (via hopping) a significant solid angle on the Bloch sphere, thus yielding an AA phase far from zero.

We further look into the sensitivity of the obtained AA phase to the exact values of T . We find that due to the hopping behavior of the cyclic states, the actual geometry of a cyclic state changes drastically as T is tuned. The resulting AA phase in general does not approach any limit. For example, Fig. 4 presents the AA phases vs T for the two cyclic states considered in Fig. 2, for a *small* time window T . It is seen that the AA phase for each individual cyclic state can be extremely sensitive to T , and oscillate violently between 0 and 2π . Based on these observations and other calculations not shown here, we conclude that the AA phase in the hopping model cannot have a large- T limit. The high sensitivity of the AA phase to T hints that the geometry of cyclic

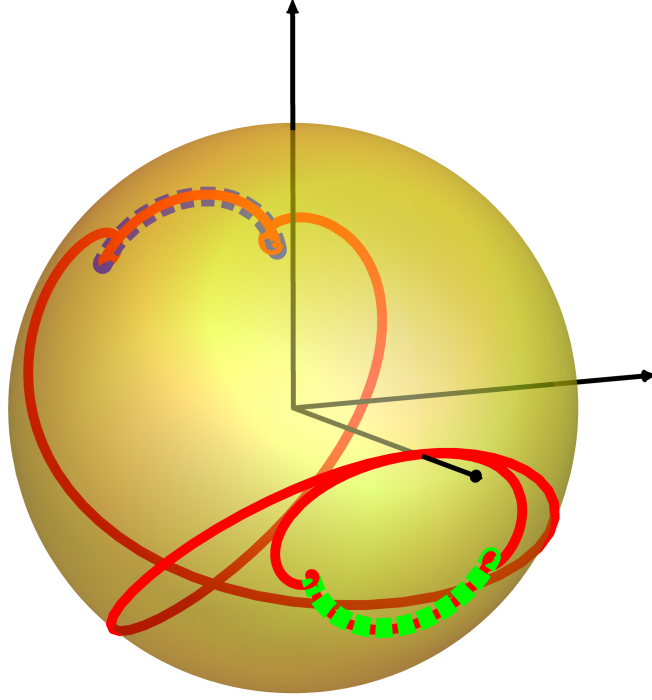


Figure 3. (color online) Geometry of one cyclic state (solid line) plotted on the Bloch sphere (one of the two considered in Fig. 2 with $\mu = 1.2$), as compared with the geometry of two instantaneous eigenstates of $H_2(t)$ (dotted lines).

states in nonunitary dynamics is both rich and fascinating.

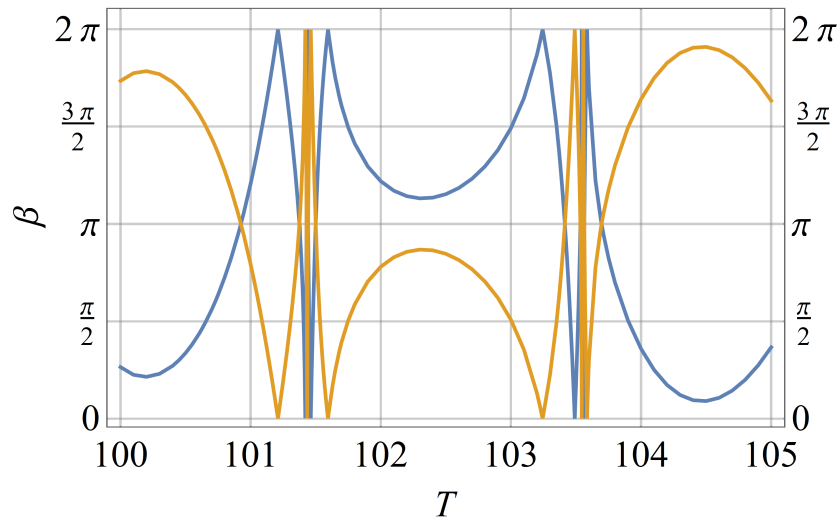


Figure 4. (color online) AA phases β as functions of period T in the hopping model $H_2(t)$ with $\mu = 1.2$. Note the small range in T and the violent oscillations in β .

5. Conclusions

Without using any sophisticated terminologies, we have shown that AA phase and likewise Berry phase in the case of adiabatic following are always real in nonunitary dynamics, thus clearing considerable confusion in the literature. Indeed, we showed that an earlier expression for AA phase in Hermitian systems can also apply to nonunitary dynamics (with normalized initial states). From these results, it becomes clear now that previous studies suggesting complex AA phases, though interesting in their own right, are not really consistent with the simple notion that the AA phase for nonunitary dynamics should just reflect the geometry of a closed curve in a projective Hilbert space.

Our fundamental understanding of the AA phase in nonunitary dynamics makes it possible to use AA phase as a quantitative tool to study the geometrical aspects of two periodically driven, non-Hermitian systems. If adiabatic following with instantaneous eigenstates presents, then the AA phase expectedly reduces to the Berry phase in the slow-driving limit. But when adiabatic following does not occur (in most cases), then the behavior of the AA phase is extremely complicated, suggesting rich geometrical features of nonunitary dynamics. This work shall motivate future studies of experimental measurements and applications of AA phases in nonunitary dynamics.

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